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SUPERSONIC FLOW OVER A WING AT HIGH ATTACK ANGLES

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The thin shock layer method [1] is applied to the problem of supersonic gas flow with Mach number $M_\infty >> 1$ over the windward surface of a thin wing. This method makes use of the significant increase in gas density at the compression discontinuity, together with the corresponding small parameter ε, which is equal to the ratio of the densities at the discontinuity. Consideration of the problem as $\varepsilon \to 0$ permits approximate consideration of the effect of the real physicochemical properties of the gas at high temperatures and determines the specifics of the problem's mathematical formulation and solution, as compared to theories in which together with the small parameter M_∞^{-1} geometric parameters are employed (attack angle α, relative wing thickness d, elongation λ), which vary over various ranges [2, 3].

If $d = O(1)$ (for example, [3-6]) or $d < O(1)$, but exceeds the compressed layer thickness in order of magnitude (for example, [7, 8]), then in the main "Newtonian" approximation the form of the discontinuity coincides with the body form, and the problem consists of finding subsequent approximations.

The most interesting and mathematically complex case is that in which the wing thickness is small and coincides in order of magnitude with the compressed layer thickness, and the form of the compression discontinuity must be determined in the process of solution. This case will be considered below. For flow over a thin wing of small elongation ($d = O(\varepsilon \tan\alpha)$, $\lambda = O(\varepsilon^{1/2} \tan\alpha)$, $\alpha = O(1)$, $\cos\alpha = O(1)$, when $\varepsilon \to 0$) the supersonic law of planar sections for thin bodies at large attack angles [2] is valid, which law in conjunction with the limiting transition $\varepsilon \to 0$ reduces the problem to calculation of a two-dimensional nonsteady-state flow in a plane perpendicular to the wing axis and moving with a velocity $V_\infty \cos\alpha$ [9, 10]. The problem of flow over a plane wing of small elongation at attack angles close to 90° ($\cos\alpha = O(\varepsilon)$) proves equivalent to the two-dimensional problem of stationary flow over a plate located perpendicular to the incident flow [11]. For the intermediate attack angle range ($\cos\alpha = O(\varepsilon^{1/2})$) such an equivalence is valid in the region adjacent to the compression discontinuity, but in the low velocity wall layer change along the chord must be considered.

For flow over a thin wing of finite extent ($d = O(\varepsilon \tan\alpha)$, $\lambda = O(1)$) at an attack angle $\alpha = O(1)$ ($\cos\alpha = O(1)$ as $\varepsilon \to 0$), the discontinuity adjoins the edge and in the fundamental approximation of the thin shock layer method the well-known law of bands is valid [12], permitting independent calculation in each plane along the wing chord of a two-dimensional flow, which in light of the unsteady state analogy [1] is equivalent to a one-dimensional unsteady

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state flow. This law is inapplicable only in narrow regions with angular size of the order of the Mach angle in the compressed layer ($\varepsilon^{1/2} \tan\alpha$) encompassing the ends of the wing or departing from cusps on the leading edge, where the three-dimensional problem must be solved [13].

The present study will consider flow over a thin wing of finite extent ($d = O(\varepsilon^{1/2})$, $\lambda = O(1)$) at large attack angles close to 90° ($\cos\alpha = O(\varepsilon^{1/2})$). In this case although the compression discontinuity is adjacent to the leading edge, or at least, the tip, the spatial flow is described by a significantly three-dimensional system of equations, which in principle distinguishes this flow regime from those studied in [9-12]. As in [5, 6, 10] an analytical solution is obtained on the basis of the fundamental property of high-density gas flows, namely the conservation of the vorticity component directed along the flow over the length of a flow line.

1. We will consider supersonic flow over a wing of finite extent at attack angles close to 90°:

$$\alpha = \pi/2 - A_*, \quad 0 < A_* \ll 1. \tag{1.1}$$

We introduce a rectangular Cartesian coordinate system Oxyz fixed to the wing with origin at the tip of the wing (see Fig. 1). We will assume the wing to be thin, and that its surface is close to the base plane y = 0, from which the attack angle is measured. We will use the thin shock layer method of [1] to solve the problem. As a result of intense gas compression in the leading compression discontinuity during supersonic flow over the windward wing surface the discontinuity surface $y = y_s(x, z)$ will also be close to the base plane, i.e., the derivatives

$$\partial y_s/\partial x,\ \partial y_s/\partial z \ll 1. \tag{1.2}$$

Assuming the gas to be ideal with constant heat capacity ratio $\varkappa$ and considering Eqs. (1.2), (1.1), we write the small parameter of the thin shock layer method, equal to the density ratio at the discontinuity, in the form

$$\varepsilon = \frac{\varkappa - 1}{\varkappa + 1} + \frac{2}{(\varkappa + 1) M_\infty^2}. \tag{1.3}$$

Transforming to the limit as $\varepsilon \to 0$ or $\varkappa \to 1$, $M_\infty \to \infty$, we will assume that

$$m = \frac{1}{2}(\varkappa - 1) M_\infty^2 = O(1). \tag{1.4}$$

To evaluate the orders of the gas dynamic function perturbations we will consider the solution of the auxiliary problem of flow over an infinite arrowlike wing, located in the plane y = 0, at the angle of attack of Eq. (1.1). The compression discontinuity attached to the leading edge is described by the equation $y_s(x, z) = Y(x - z\tan\Lambda)$, where Λ is the sagittal angle, $\tan\Lambda = O(1)$; $Y \ll 1$ is the still undefined scale of the derivatives $\partial y_x/\partial x = Y$, $\partial y_s/\partial z = -Y\tan\Lambda$. Using the relationship on the compression discontinuity, given conditions (1.1)-(1.4), we find an expression for the vertical component of the velocity v, which in view of the impermeability of the wing vanishes. Hence, considering only terms of the least order of smallness, we obtain

$$v/V_\infty = A_* Y - Y^2 - Y^2 \operatorname{tg}^2\Lambda - \varepsilon + \ldots = 0. \tag{1.5}$$

The most general case is that in which all terms of Eq. (1.5) are of identical order of smallness as $\varepsilon \to 0$. Therefore, we take

$$Y \sim A_* \sim \varepsilon^{1/2}. \tag{1.6}$$

The quantity $A = A_*/\varepsilon^{1/2} = (\pi/2 - \alpha)\varepsilon^{1/2}$ will be the similarity parameter. The solution of quadratic equation (1.5) corresponding to the weak branch of the discontinuity has the form

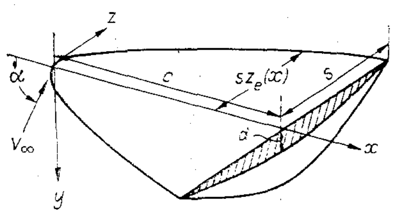

Fig. 1

$$Y/\varepsilon^{1/2} = (1/2)[A - \sqrt{A^2 - 4(1 + \mathrm{tg}^2\,\Lambda)}\,]\cos^2\Lambda.$$

The flow regime with discontinuity attached to edge is realized for $A \geqslant 2$ and saggital angles $\Lambda \leqslant \arctan((1/2)\sqrt{A^2-4})$.

From the conditions on the discontinuity with consideration of Eq. (1.6) we find the order of the components of the velocity, the pressure, and density,

$$u \sim w \sim \varepsilon^{1/2}V_\infty, \quad v \sim \varepsilon V_\infty, \quad (p - p_\infty)/(\rho_\infty V_\infty^2) - 1 \sim \varepsilon, \quad \rho_\infty/\rho \sim \varepsilon. \tag{1.7}$$

2. We will now consider the fundamental problem of flow over the windward surface of a wing at large angles of attack. To study the flow structure in the compressed layer we introduce dimensionless variables having the order of unity as $\varepsilon \to 0$:

$$x^0 = x/c, \; y^0 = y/c\varepsilon^{1/2}, \; z^0 = z/c, \tag{2.1}$$

where c is the characteristic longitudinal dimension (see Fig. 1). The wing thickness, measured from the base plane $y = 0$, will be assumed to be of the same order of magnitude as the thickness of the shock layer: $d \sim c\varepsilon^{1/2}$. The equation of the wing surface has the form

$$y_b = d_{\max} f(x^0, z^0). \tag{2.2}$$

We specify the equation of the projection of the wing's leading edge on the base plane in the form $|z| = sz_e(x)$. We will assume that the estimated values of Eq. (1.7) are valid over the entire compressed layer and introduce the following expansions of the unknown variables:

$$\begin{gathered} u/V_\infty = \varepsilon^{1/2}u^0(x^0, y^0, z^0) + \ldots, \; v/V_\infty = \varepsilon v^0(x^0, y^0, z^0) + \ldots, \\ w/V_\infty = \varepsilon^{1/2}w^0(x^0, y^0, z^0) + \ldots, \quad (p - p_\infty)/(\rho_\infty V_\infty^2) = 1 + \varepsilon p^0(x^0, y^0, z^0) + \ldots, \\ \rho_\infty/\rho = \varepsilon - \varepsilon^2(1 + p^0) - [m\varepsilon^2/(m+1)](u^{02} + w^{02}) + \ldots, \\ y_s = c\varepsilon^{1/2}S(x^0, z^0) + \ldots, \; y_b = c\varepsilon^{1/2}F(x^0, z^0). \end{gathered} \tag{2.3}$$

Substitution of Eqs. (2.1)-(2.3) in the exact gas-dynamics equations and boundary conditions on the compression discontinuity and wing surface in the limit $\varepsilon \to 0$ with conditions (1.1), (1.4), (1.6) in the fundamental approximation leads to formulation of the following problem (superscripts are omitted):

$$u_x + v_y + w_z = 0; \tag{2.4}$$

$$uu_x + vu_y + wu_z = 0; \tag{2.5}$$

$$uw_x + vw_y + ww_z = 0; \tag{2.6}$$

$$uv_x + vv_y + wv_z = -p_y; \tag{2.7}$$

$$\begin{gathered} u_s = A - S_x, \quad w_s = -S_z, \quad v_s = AS_x - S_x^2 - S_z^2 - 1, \quad p_s = 2AS_x - \\ - S_x^2 - S_z^2 - 1 - A^2; \end{gathered} \tag{2.8}$$

$$\begin{gathered} v_b = u_bF_x + w_bF_z, \; F = \Delta f, \; 0 \leqslant x \leqslant 1, \\ |z| \leqslant \Omega z_e(x) \quad (z_e(1) = 1), \end{gathered} \tag{2.9}$$

where $u_s \equiv u[x, S(x, z), z]$; $u_b \equiv u[x, F(x, z), z]$, etc. In the fundamental approximation of the thin shock layer method flow over a wing of finite extent at high attack angles is described by a three-dimensional nonlinear system of equations. From Eqs. (2.4)-(2.9) we have the similarity law: For flow over wings of similar form at high attack angles the dimensionless gas-dynamic functions denoted by the superscript 0 in Eq. (2.3) depend on the dimensionless coordinates of Eq. (2.1) and the similarity parameters:

$$A = (\pi/2 - \alpha)/\varepsilon^{1/2}, \; \Delta = d_{\max}/c\varepsilon^{1/2}, \; \Omega = s/c,$$

where 2s is the extent of the wing.

For the normal wing force coefficient c_N (without consideration of the negligible contribution of the downwind surface) we have the similarity law

$$(c_N - 2 - 2/\varkappa M_\infty^2)/\varepsilon = C(A, \Delta, \Omega), \text{ where } C = 2\int_0^1\int_0^{\Omega z_e} p\,dx\,dz.$$

In addition, the correction to the constant density value $\rho = \varepsilon^{-1}\rho_\infty$ also depends on the parameter m of Eq. (1.4).

Boundary problem (2.4)-(2.9) describes flow in the entire compressed layer from the compression discontinuity to the wing tip. The discontinuity may be attached to the leading edge or detached. With consideration of thickness, in analogy to section 1 we obtain a condition for existence of a discontinuity attached to the edge in the form

$$[F_z^e - (A - F_x^e)\,\Omega z_e']^2 \geqslant 4(1 + \Omega^2 z_e'^2),$$

where $F_x^e \equiv F_x[x, \Omega z_e(x)]$; $F_z^e \equiv F_x$ x, $\Omega z_e(x)]$.

In the case in which the discontinuity is detached from the surface, as in [11], a region of slow flow appears near the surface, where the order of magnitude of the velocities differs from Eq. (1.7) and which must be analyzed separately. The thickness of the shock layer, determined by matching the solution in this region with the external region in the main portion of the layer, will be somewhat larger than $\varepsilon^{1/2}$ in order of magnitude.

3. We will note a number of properties of flows described by Eqs. (2.4)-(2.7) which will be of further use. We introduce notation for an operator indicating differentiation along a flow line

$$D \equiv u\partial/\partial x + v\partial/\partial y + w\partial/\partial z.$$

Equations (2.5), (2.6), having the form Du = Dw = 0, show that the longitudinal and lateral velocity components are constant along the flow lines. It follows from this that the projections of the flow lines onto the base plane y = 0 are straight lines with a slope dz/dx = w/u, i.e., the flow lines are plane curves lying in planes orthogonal to the base plane.

The expression of the component of the vorticity in the direction of the flow has the form

$$\omega_v = \frac{u^2}{\sqrt{u^2 + u^2}}\left(\frac{w}{u}\right)_y + \ldots$$

Equations (2.4)-(2.6) show that in the basic approximation of the thin shock layer method the vorticity component directed along the flow is constant along the flow lines. Consequently,

$$D[(w/u)_y] = 0. \tag{3.1}$$

This fundamental conservation property will be used in the future for integration of system (2.4)-(2.7).

We will study the characteristic properties of this system. Let $\Phi(x, y, z)$ = const be the equation of the characteristic surface. Writing the equations of the characteristics, we obtain $\Phi_y^2(D\varphi)^2 = 0$. Therefore the characteristic properties will be manifested by flow lines (surfaces) and cylindrical surfaces $\Phi(x, z)$ = const, orthogonal to the base plane.

4. The nonlinear system of equations in partial derivatives, Eqs. (2.4)-(2.7), ca be integrated and a solution of the problem of Eqs. (2.4)-(2.9) can be obtained in the form of analytical expressions of the gas-dynamics functions in terms of the form of the compression discontinuity. To do this, in place of continuity equation (2.4) we use the equivalent equation of conservation of the vorticity component along the flow (3.1) and transform to characteristic coordinates x, ψ, z, where ψ is the flow line function $D\psi = 0$. In accordance with the form of Eqs. (2.5), (2.6), (3.1) we take $\psi = w/u$. The system (2.4)-(2.7) takes on the form

$$(y_\psi)_x + \psi(y_\psi)_z = 0; \tag{4.1}$$

$$u_x + \psi u_z = 0; \tag{4.2}$$

$$v = u(y_x + \psi y_z); \tag{4.3}$$

$$p_\psi = -u y_\psi\,(v_x + \psi v_z). \tag{4.4}$$

In the new variables the quantity y has become an unknown function, defined by the second-order equation (4.1). Integrating, we find $y_\psi = \Gamma(\psi, z - \psi x)$. Here the function $\Gamma^{-1} = \omega_e\sqrt{1 + \psi^2}$ characterizes the distribution of the flow component of vorticity. We note that $D(z - \psi x) = 0$, i.e., the quantity $\theta = z - \psi x$ together with ψ are functions of the flow lines. A second integration with consideration of conditions on the wing y = F(x, z) with $\psi = \psi_b(x, z)$ gives

$$y = F(x, z) + \int_{\psi_b}^{\psi} \Gamma(\psi', z - \psi' x)\, d\psi'. \tag{4.5}$$

From Eqs. (4.2)-(4.4) we obtain

$$u = U(\psi, z - \psi x), \qquad w = \psi U(\psi, z - \psi x),$$
$$v = U\left\{F_x + \psi F_z + \int_{\psi_b}^{\psi} (\psi - \psi')\, \Gamma_{\theta'} d\psi' - \Gamma_b\left[(\psi_b)_x + \psi(\psi_b)_z\right]\right\}. \tag{4.6}$$

To satisfy impermeability condition (2.9) we require that one of the following equalities be satisfied: $\Gamma_b = 0$ or $(\psi_b)_x + \psi_b(\psi_b)_z = 0$. On the compression discontinuity the function $\psi = \Psi$, while, according to Eq. (2.8), $\Psi = S_z(S_x - A)^{-1}$. Satisfying condition (2.8) for v, which has the form of Eq. (4.6), we find the form of the function Γ on the discontinuity

$$\Gamma_s(x, z) = [\Psi(x, z)(S_{xx} - S_{zz}) - (1 - \Psi^2)S_{xz}]^{-1}. \tag{4.7}$$

The pressure distribution is determined from Eqs. (4.4), (4.6) and has the form

$$p = p_s + \int_{\psi}^{\Psi} (v_x + \psi' v_z)\, \Gamma(\psi', z - \psi' x)\, d\psi'.$$

We denote by $\chi(\Psi, \Theta)$, $\nu(\Psi, \Theta)$ the abscissa and applicate of the point of entry into the shock layer of the flow line along which $\psi = \Psi$, $\theta = \Theta$. For determination of χ, ν we use the equations $\Psi = S_z(\chi, \Psi\chi + \Theta)/[S_x(\chi, \Psi\chi + \Theta) - A]$, $\nu = \Theta + \Psi\chi(\Psi, \Theta)$. Then the functions U, Γ in the flow field are found from their values directly behind the discontinuity

$$U(\psi, z - \psi x) = u_s(\chi, \nu), \; \Gamma(\psi, z - \psi x) = \Gamma_s(\chi, \nu), \tag{4.8}$$

where $\chi = \chi(\psi, z - \psi x)$; $\nu = z - \psi(x - \chi)$.

Finally, we obtain an analytical solution of the problem, representing the gas dynamic functions in the form of quadratures and functional dependences in terms of the form of the compression discontinuity S and the function Γ, which, according to Eqs. (4.5), (4.7), (4.8), satisfy the system of equations

$$S(x, z) = F(x, z) + \int_{\psi_b}^{\Psi} \Gamma(\psi, z - \psi x)\, d\psi, \quad \Psi = S_z(S_x - A)^{-1},$$
$$\Gamma(\Psi, z - \Psi x) = [\Psi(S_{xx} - S_{zz}) - (1 - \Psi^2)S_{xz}]^{-1}. \tag{4.9}$$

The form of the function $\psi_b(x, z)$, which has straight level lines, depends on the regime of flow over the leading edge of the wing. The following cases can be distinguished.

A. In the vicinity of the sharp leading edge the windward surface has a form smooth in plan (i.e., $z_e'(0) = \infty$; F_x^e, F_z^e are continuous for $y > F_e(x) \equiv F[x, \Omega z_e(x)]$), and the discontinuity is attached to the edge. Then on the edge $\psi_e = w_e/u_e$, and the solution of the flow problem in the vicinity of the leading edge with the aid of power series analogous to those of [14] allows us to obtain

$$u_e = A - F_x^e - z_e'\Omega\left(F_z^e + w_e\right),$$
$$w_e(x) = \frac{\Omega z_e'\left(A - F_x^e\right) - F_z^e\left(1 + 2\Omega^2 z_e'^2\right) - \sqrt{\left[F_z^e + \left(A - F_x^e\right)\Omega z_e'\right]^2 - 4\left(1 + \Omega^2 z_e'^2\right)}}{2\left(1 + \Omega^2 z_e'^2\right)}. \tag{4.10}$$

Simultaneously, the form of the function Γ is determined on the leading edge. Thus, for a plane wing

$$\Gamma_e^{-1}(x) = \frac{2A\Omega^2 w_e^3 z_e' z_e''\left[A\left(1 + w_e^2\right) - 2\Omega w_e z_e'\right]}{\left(1 + w_e^2\right)\left[\Omega^2 z_e'^2\left(A^2 - 4\right) - 4\right]}. \tag{4.11}$$

For a wing with thickness considered the expression for Γ_e is very cumbersome and will not be presented. On the wing surface the functions Ψ_b, Γ_b are expressed in the form

$$\psi_b(x, z) = \psi_e(\chi_e), \; \Gamma_b(x, z) = \Gamma_e(\chi_e), \tag{4.12}$$

where $\chi_e(x, z)$ is the abscissa of the point of intersection of the flow line located on the wing with the leading edge, defined as the root of the functional equation

$$z - \Omega z_e(\chi_e) = \psi_e(\chi_e)(x - \chi_e). \tag{4.13}$$

B. The discontinuity is attached to a leading edge having a cusp at the tip: $\psi_e(+0) = -\psi_e(-0) \neq 0$. In this case in the central portion of the wing for $|z| \leq \psi_e(+0)x$ there is a sheaf of low lines which pass through the tip $x = z = 0$. Hence for these $\chi_e = 0$, $\theta_b = z - \psi_b x = 0$, $\psi_b = z/x$. Moreover, $\Gamma_b = 0$, since the curvature of the discontinuity is singular at the tip. In the cantilevered portions of the wing at $\psi_e(+0)x < |z| < \Omega z_e(x)$ the functions χ_e, ψ_b, Γ_b are defined by Eqs. (4.10)-(4.13).

C. The discontinuity is attached to the wing only at the tip and detached from the edge. Then all flow lines on the wing pass through the tip and on the entire wing surface $\psi_b = z/x$, $\chi_e = \Gamma_b = 0$.

We also note that the denominator in Eq. (4.11) vanishes at the point where the discontinuity detaches from the edge, where $|z_e| = 2/\Omega\sqrt{A^2 - 4}$. Consequently, in the vicinity of this point there is intense formation of vorticity oriented along the flow.

5. If we transform to new independent variables x, χ, z [5], then the function Γ is eliminated from the solution. Thus, system (4.9) for the form of the discontinuity takes on the form

$$S(x, z) = F(x, z) + \int\limits_{\chi_e}^{x} \frac{d\chi}{u_s(\chi, \nu)\,[1 + (x - \chi)\,\Psi_z(\chi, \nu)]}, \tag{5.1}$$

$u_s = A - S_x$, $\Psi = -S_z/u_s$, $\nu = z - \Psi(\chi, \nu)(x - \chi)$. As an example we will consider the solution of the converse problem with specified compression discontinuity surface of conical form in the vicinity of the symmetry plane $z = 0$. We introduce the conical variables $\eta = y/x$, $\zeta = z/x$. At $\zeta \ll 1$ we have

$$\eta_s = S_0 - \frac{1}{2} S_2\zeta^2 - \frac{1}{4} S_4\zeta^4 + \ldots,$$

where S_0, S_2, S_4, ... are specified constant coefficients. Solution of system (5.1) reveals that the form of the body corresponding to the conical discontinuity is also conical and

$\eta_b = F_0 - \frac{1}{2}F_2\zeta^2 - \frac{1}{4} F_4\zeta^4 + \ldots,$ where $F_0 = S_0 - \dfrac{(A - S_0 - S_2) + S_2 \ln\dfrac{S_2}{A - S_0}}{(A - S_0 - S_2)^2}$. By integrating it is simple to obtain subsequent coefficients F_2, F_4, ..., which are expressed in terms of S_0, S_2, S_4, ... For example, to calculate the coefficient F_2, aside from S_0, S_2, it is necessary to specify S_4 in the next term of the expansion of the discontinuity form.

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THREE-DIMENSIONAL DIFFUSIVE BOUNDARY-LAYER PROBLEMS

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UDC 532.72

1. PROBLEM FORMULATION. CHOICE OF COORDINATE SYSTEM

Consider three-dimensional viscous incompressible laminar flow past a solid or liquid particle of arbitrary shape with convective diffusion to the surface. It is assumed that the Peclet number $\mathrm{Pe} = aUD^{-1}$ is large; here a is the characteristic dimension of the particle (it is usually the radius of an equivalent sphere by volume), U is the characteristic flow velocity (at infinity), D is the diffusion coefficient. It is also assumed that concentration C_* is constant at the surface and away from it, equal to C_s and C_∞, respectively, and the flow field is determined from the solution of the corresponding hydrodynamic problem of the flow past the particle.

Orthogonal curvilinear coordinate system in ξ, η, λ connected to the body surface and streamlines is used in the analysis as in [1, 2]. The directions of unit vectors at any point M in the surrounding fluid are given by e_ξ, e_η, e_λ (Fig. 1). The unit vector e_ξ is determined by the direction of the normal to surface of the particle passing through the point M; the unit vector e_η is given by the direction of the projection of the velocity vector at the point M on the plane perpendicular to e_ξ; the unit vector e_λ is chosen such that the system of unit vectors e_ξ, e_η, e_λ is a right-handed orthogonal triad (Fig. 1). The origin of the coordinate system and the procedure for computing curvilinear coordinates (i.e., the dependence of metric tensor components $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\lambda\lambda}$ on ξ, η, λ), are chosen from the point of view of convenience in each particular case; for concreteness, we further assume that the surface of the particle is given by a fixed value ξ = 0. In such a coordinate system the fluid velocity vector at each point is given by $v = \{v_\xi, v_\eta, 0\}$.

The equation of continuity for an incompressible fluid has the form

$$\operatorname{div} \mathbf{v} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\xi}\left(v_\xi\sqrt{\frac{g}{g_{\xi\xi}}}\right) + \frac{\partial}{\partial\eta}\left(v_\eta\sqrt{\frac{g}{g_{\eta\eta}}}\right)\right] = 0. \tag{1.1}$$

The function ψ(ξ, η, λ) is determined as the solution to the system

$$\frac{\partial\psi}{\partial\xi} = v_\eta\sqrt{\frac{g}{g_{\eta\eta}}},\quad \frac{\partial\psi}{\partial\eta} = -v_\xi\sqrt{\frac{g}{g_{\xi\xi}}}. \tag{1.2}$$

Then the equation of continuity (1.1), which coincides with the condition for integrability of the system (1.2), is automatically satisfied. The constant of the integration in Eq. (1.2) is chosen such that the function ψ becomes zero at the surface.

The surface ψ(ξ, η, λ) = const wholly consists of streamlines. The function ψ has a simple physical meaning: It is the three-dimensional analog of stream function. In the plane and axisymmetric cases ψ coincides with stream function.

In nondimensional variables the equation of stationary convective diffusion and boundary conditions in curvilinear coordinate system ξ, η, λ are written in the following form using Eq. (1.2):

$$-\frac{\partial(c,\psi)}{\partial(\xi,\eta)} = \frac{1}{\mathrm{Pe}}\left\{\frac{\partial}{\partial\xi}\left(\frac{\sqrt{g}}{g_{\xi\xi}}\frac{\partial c}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{\sqrt{g}}{g_{\eta\eta}}\frac{\partial c}{\partial\eta}\right) + \frac{\partial}{\partial\lambda}\left(\frac{\sqrt{g}}{g_{\lambda\lambda}}\frac{\partial c}{\partial\lambda}\right)\right\}; \tag{1.3}$$

$$\xi = 0,\ c = 0;\ \xi \to \infty,\ c \to 1, \tag{1.4}$$

$$c = (C_* - C_s)/(C_\infty - C_s),\quad \mathrm{Pe} = aU/D,\quad g = g_{\xi\xi}g_{\eta\eta}g_{\lambda\lambda},$$

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